(*b*) Let $x \equiv l - R$. If the pendulum beats seconds show that x must be a root of the cubic $a_0x^3 + a_1x^2 + a_2x + a_3 = 0$, where

$$a_0 \equiv 80\pi^3r^2,$$

$$a_1 \equiv 40\pi[4\pi^2r^3 - 3r^2(g - 2\pi^2R) + 4\pi^2(R^2 - r^2)^{3/2}],$$

$$a_2 \equiv - 20\pi\{4(g - 2\pi^2R)[2R^3 + 3r^2R + 2(R^2 - r^2)^{3/2}] + 3\pi^2r^4\},$$

$$\begin{aligned} a_3 \equiv & - [80\pi^3r^2(h - a)^3 - 120\pi gr^2(h - a)^2 + 60\pi^3r^4(h - a) - 5\pi^2abe(12a^2 + b^2) \\ & - 20\pi^2htw(12a^2 - 12ah + 4h^2 + w^2) - 80ga^2be + 120ghtw(h - 2a) - 224\pi^3R^5 \\ & + 160\pi gR^4 - 320\pi^3r^2R^3 + 240\pi gr^2R^2 + 60\pi^3r^4R - 60\pi gr^4 - 16\pi(14\pi^2R^2 + \pi^2r^2 \\ & - 10gR)(R^2 - r^2)^{3/2}]. \end{aligned}$$

(*c*) Given $a = 1$, $b = 0\cdot 8$, $l = 2$, $g = 980\cdot 5$, $h = 2$, $P_0 = 2$, $r = 0\cdot 49/4$, $R = 12\cdot 01/4$, $t = 0\cdot 64$, and $w = 1\cdot 2$; calculate l correctly to four, or more, decimal places.

(*d*) Compute the percentage error affecting the result if g were calculated from the true value of the period (2 secs), as obtained experimentally, while neglecting all moments of forces and of inertia except those pertaining to the spherical bob.

2744. Proposed by E. B. ESCOTT, Chicago, Ill.

An insurance company computes its quarterly premiums by adding 6 per cent. to the annual premium and dividing by 4. If a policyholder pays quarterly, what rate of interest is he paying?

2745. Proposed by G. I. HOPKINS, Manchester, N. H.

A recent English publication contains the following method of constructing a regular polygon of 17 sides: Draw the radius CB perpendicular to the diameter AQ of the circle whose center is B. On BC lay off BD equal to one-fourth of BC. On BA, lay off BE and draw DE making angle BDE one-fourth of angle BDA. On BQ lay off BF and draw DF making angle FDE 45°. On AF as diameter, construct semi-circle FRA intersecting CB in H. With E as center and EH as radius construct semi-circle LHK intersecting CB in H. At the points L and K draw the ordinates NL and MK. Bisect the arc NM and let P be the point of bisection. Then the chord $NP(= MP)$ is a side of the regular polygon of 17 sides. Is the method of construction correct?

SOLUTIONS OF PROBLEMS.

433 (Calculus). Proposed by LOUIS O'SHAUGHNESSY, University of Pennsylvania.

Solve the differential equation, $\dfrac{d^{1/2}y}{dx^{1/2}} - \dfrac{y}{x} = 0$.

DISCUSSION BY EMIL L. POST, Columbia University.

In the April number of the MONTHLY appeared two solutions of the fractional differential equation

$$\frac{d^{1/2}y}{dx^{1/2}} = \frac{y}{x}. \tag{1}$$

The first of these, $y = Cx^{-1/2}e^{-1/x}$, was obtained by reducing the given equation to an ordinary differential equation of the first order; while the second, $y = A_0(1 - i\sqrt{\pi}x^{-1/2} - 2x^{-1} + i\sqrt{\pi}x^{-3/2} + \ldots)$ was found by equating coefficients in an assumed expansion in series. Now each of the two methods used would seem to indicate that only one solution was possible, *i. e.*, the solution found by the corresponding method, yet the two solutions are clearly irreducible. A discussion of this difficulty might serve as a beginning of that more thorough discussion of the subject which the proposer of the problem desires.

Now first of all what does $d^{1/2}y/dx^{1/2}$ mean? More generally, what does $d^\mu y/dx^\mu$ mean where μ is any number? Before 1860, certainly, the method followed in answering that question was that due to Liouville. Starting from the known fact that $(d^\mu/dx^\mu)e^{ax} = a^\mu e^{ax}$, when μ was a positive integer, and also when μ was a negative integer, if by $d^{-p}\cdot y/dx^{-p}$ we mean the pth indefinite integral of y, he then assumed the relation to hold for all values of μ, and proceeded from that to obtain everything else. The most important formula that thus resulted was

$$\frac{d^\mu x^n}{dx^\mu} = (-1)^\mu \frac{\Gamma(-n+\mu)}{\Gamma(-n)} x^{n-\mu}. \tag{2}$$

It was this formula that the proposer of the problem used when he solved it by expansion in a series of powers of x.

But this method was too narrow, when compared with the general and rigorous methods of analysis then in use, to last. It was Riemann who first gave a definition of $d^\mu y/dx^\mu$ essentially like those used now. He gave it in the form of a definite integral. More recent writers express it as a contour integral. One of the most significant features of this newer development is the introduction of limits of "differentiation" thereby making fractional derivatives more like definite integrals than ordinary derivatives, while the ordinary derivative appears as a peculiar (though singularly important!) degeneration. The analogy of the general binomial expansion expressed as an infinite series, with its particular terminating form for a positive integral power is complete.

A useful form applicable only when the real part of the index of differentiation is negative is

$$\left\{\frac{d^{-\mu}}{dx^{-\mu}}\right\}_{x_0}^{X} f(x) = \frac{1}{\Gamma(\mu)}\int_{x_0}^{X} (X-x)^{\mu-1} f(x)\,dx. \tag{3}$$

It is to be noticed that when $x_0 = 0$, it becomes Riemann's form, while for $x_0 = -\infty$, it gives all the consistent results obtained by Liouville's method. As it stands it is more general than either.

The theorem that justifies the definition is that

$$\left\{\frac{d^{\mu_1}}{dx^{\mu_1}}\right\}_{x_0}^{X}\left\{\frac{d^{\mu_2}}{dy^{\mu_2}}\right\}_{x_0}^{X} f(y) = \left\{\frac{d^{\mu_1+\mu_2}}{dx^{\mu_1+\mu_2}}\right\}_{x_0}^{X} f(x), \tag{4}$$

where for positive integral indices we get the ordinary derivatives. The generalization of Leibnitz's theorem also follows, *i, e.*,

$$\left\{\frac{d^{\mu}}{dx^{\mu}}\right\}_{x_0}^{X} u\cdot v = u\left\{\frac{d^{\mu}}{dx^{\mu}}\right\}_{x_0}^{X} v + \mu\frac{du}{dx}\left\{\frac{d^{\mu-1}}{dx^{\mu-1}}\right\}_{x_0}^{X} v + \frac{m(\mu-1)}{2^3}\frac{d^2u}{dx^2}\left\{\frac{d^{\mu-2}}{dx^{\mu-2}}\right\}_{x_0}^{X} v + \cdots. \tag{5}$$

We are now in a position to reconcile the two solutions of equation (1). In reducing (1) to an equation of the first order, we used equation (5), and then equation (4) in the form

$$\frac{d^{1/2}}{dx^{1/2}}\frac{d^{1/2}y}{dx^{1/2}} = \frac{dy}{dx},$$

and

$$\frac{d^{-1/2}}{dx^{-1/2}}\frac{d^{1/2}y}{dx^{1/2}} = y.$$

Now although this procedure is valid when the lower limit, x_0, is finite, it is not valid when $x_0 = -\infty$, for we then have

$$\frac{d^{-1/2}}{dx^{-1/2}}\frac{d^{1/2}y}{dx^{1/2}} = y - B \tag{6}$$

where B is a constant depending on y.[1] The second solution can be now obtained in the same ways as the first.

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[1] The following explains the failure of the general theorem. From (3), when extended for all values, we find

$$\left\{\frac{d^{\mu}}{dx^{\mu}}\right\}_{x_0}^{X} B = B\,\frac{(X-x_0)^{-\mu}}{\Gamma(1-\mu)}\,.$$

If x_0 is not infinite on taking the μth derivative of this result, we get back C_1B. If x_0 is infinite, the above result is zero, the $-\mu$th derivative leaves it zero, and so the constant is lost, as (6) indicates.

Clearing (1) of fractions, operating through by $d^{1/2}/dx^{1/2}$ and using (5) we find

$$x\frac{d^{1/2}}{dx^{1/2}}\frac{d^{1/2}y}{dx^{1/2}}+\frac{1}{2}\frac{d^{-1/2}}{dx^{-1/2}}\frac{d^{1/2}y}{dx^{1/2}}=\frac{d^{1/2}y}{dx^{1/2}}$$

Using (4), (6), and (1) this reduces to

$$x\frac{dy}{dx}+\tfrac{1}{2}(y-B)=\frac{y}{x}$$

whose solution is

$$y=x^{-1/2}e^{-1/x}\left[\frac{B}{2}\int x^{-1/2}e^{1/x}dx+C\right]. \tag{7}$$

Since the order of the equation solved was raised, the best we can say is that if a solution of (1) exists for $x_0=-\infty$ it is contained in (7). As a check, we expand (7) in series obtaining

$$y=B+Cx^{-1/2}-2Bx^{-1}-Cx^{-3/2}+\cdots,$$

which, on comparison with the known solution for $x_0=-\infty$, gives $B=A_0$; $C=-i\sqrt{\pi}A_0$. Since C is no longer arbitrary, we must change the indefinite integral of equation (7) to a definite integral. We then find,

$$y=A_0\left[1-i\sqrt{\pi}x^{-1/2}e^{-1/x}+x^{-1/2}e^{-1/x}\int_{-\infty}^{x}t^{-3/2}e^{1/t}dt\right], \tag{8}$$

which is valid for all except positive real values of x.

The derivation of the solution

$$y=Cx^{-1/2}e^{-1/x}, \tag{9}$$

which was obtained in the same manner as (7) except that $B=0$, was not rigorous since it was not checked. If equation (1) be written

$$y=\frac{d^{-1/2}}{dx^{-1/2}}\left(\frac{y}{x}\right)$$

we can check directly by the use of (3). In fact

$$\frac{d^{-1/2}}{dx^{-1/2}}\left(\frac{y}{x}\right)=\frac{c}{\Gamma(\frac{1}{2})}\int_{x_0}^{X}(X-x)^{-1/2}x^{-3/2}e^{-1/x}dx=\frac{c}{\Gamma(\frac{1}{2})}X^{-1/2}e^{-1/X}\int_{0}^{(1/x_0-1/X)}t^{-1/2}e^{-t}dt$$

which is obtained by letting $x=X/(1+tX)$.

Since $\int_0^\infty t^{-1/2}e^{-t}dt=\Gamma(\frac{1}{2})$, if we let $x_0=0$ we find

$$\frac{d^{-1/2}}{dx^{-1/2}}\left(\frac{y}{x}\right)=CX^{-1/2}e^{-1/X}=y$$

which checks the solution. Clearly $\{d^{-1/2}/dx^{-1/2}\}_0{}^X(y/x)$ exists only when the real part of X is positive. The two solutions thus supplement each other.

Suppose now that $x_0\neq 0$, and yet is finite. Is there no solution? Clearly we always have the trivial solution $y=0$. If however we restrict ourselves to real values of the variable and let $x_0<0$, the function $y=g(x)$ where $g(x)=0$ for $x_0\leq x\leq 0$, and $g(x)=cx^{-1/2}e^{-1/x}$ for $x>0$ clearly satisfies the auxiliary differential equation, and also the original one; since

$$\left\{\frac{d^{-1/2}}{dx^{-1/2}}\right\}_{x_0}^{x}\left(\frac{g(x)}{x}\right)=\left\{\frac{d^{-1/2}}{dx^{-1/2}}\right\}_{x_0}^{x}cx^{-3/2}e^{-1/x}=g(x);\ x>0,$$

$$\left\{\frac{d^{-1/2}}{dx^{-1/2}}\right\}_{x_0}^{x}\left(\frac{g(x)}{x}\right)=0=g(x);\ x\leq 0.$$

However, (8) and (9) are the only analytic solutions here found. It is noteworthy that they correspond to the definition of the "generalized derivative" given by Liouville and Riemann respectively.